Basic Concepts of Differential Algebra

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1. Basics
   - Differential Fields
   - Integration of Rational Functions
   - Rothstein/Trager Method (rational function case)
2. Algebraic Integration
   - Elementary Functions
   - Liouville's Principle
   - The Risch Algorithm
3. Application
   - Special Systems of Linear ODEs

What are we talking about?

The Problem

Given $f(x)$, find $g(x)$ such that

$$g'(x) = f(x)$$

Examples:

$$\int 3x^2 + 2x + 1 \, \mathrm{dx} = ?$$

$$\int \frac{3x^2 + 2x + 1}{5x^3 + 4x^2 + 3x + 2} \, \mathrm{dx} = ?$$

$$\int \frac{x}{\exp(x) + 1} \, \mathrm{dx} = ?$$

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Definitions

Definition (Differential Field)

A field F (char(F) = 0) with mapping $D : F \to F$ such that $\forall\ f, g \in F$:

$$D(f+g) = D(f) + D(g)$$
$$D(f \cdot g) = f \cdot D(G) + g \cdot D(f).$$

D is called differential operator.

Definition (Field of Constants)

Let F be a differential field, D a differential operator. The *field of constants* K is a subfield of F defined by

$$K = \{c \in F : D(c) = 0\}$$

More Definitions

Definition (Differential Extension Field)

Let F, G be differential fields, D_F, D_G differential operators. Then G is a *differential extension field* of F if G is extension field of F and

$$D_F(f) = D_G(f) \quad \forall\, f \in F.$$

Definition (Logarithmic Functions)

Let F be a differential field and G be a differential extension field of F. Then $\theta \in G$ is called *logarithmic* over F if there exists $u \in F$ such that

$$D(\theta) = \frac{D(u)}{u}.$$

Write $\theta = \log(u)$.

Rational Part of the Integral: Hermite's Method

Problem:

Given $a/b \in K(x)$ determine $I \in K^*(x)$ such that $\int a/b = I$.

Hermite's Method

- apply Euclidean division, normalize:

$$\int \frac{a}{b} = \int p + \int \frac{r}{q}$$

- compute square-free factorization of q:

$$q = \prod_{i=1}^{k} q_i^i$$

- compute partial fraction expansion of r/q:

$$\frac{r}{q} = \sum_{i=1}^{k} \sum_{j=1}^{i} \frac{r_{ij}}{q_i^j} \quad \text{s.t. } \deg(r_{ij}) < \deg(q_i)$$

Hermite's Method (cont'd)

We have:

$$\int \frac{r}{q} = \sum_{i=1}^{k} \sum_{j=1}^{i} \int \frac{r_{ij}}{q_i^j}.$$

q_i square-free $\Leftrightarrow \gcd(q_i, q_i') = 1$

$\rightarrow s \cdot q_i + t \cdot q_i' = r_{ij}$ (extended Euclidean algorithm)

$$\int \frac{r_{ij}}{q_i^j} = \int \frac{s}{q_i^{j-1}} + \int \frac{tq_i'}{q_i^j}.$$

Integration by Parts:

$$\int \frac{tq_i'}{q_i^{\mathbf{j}}} = \frac{-t/(j-1)}{q_i^{j-1}} + \int \frac{t'/(j-1)}{q_i^{\mathbf{j-1}}}.$$

Where are we?

1. Problem:

$$\int \frac{a}{b} = ?$$

2. Euclidean Division:

$$\int \frac{a}{b} = \int p + \int \frac{r}{q}$$

3. Partial Fraction Expansion:

$$\int \frac{a}{b} = \int p + \sum_{i=1}^{k} \sum_{j=1}^{i} \int \frac{r_{ij}}{q_i^j}$$

4. Integration by Parts:

$$\int \frac{a}{b} = \int p + \frac{c}{d} + \sum_{i=1}^{k} \int \frac{r_i}{q_i}$$

with $\deg(r_i) < \deg(q_i)$, q_i square-free

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Logarithmic Part of the Integral

Let $a, b \in K[x]$, b square-free, $\deg(a) < \deg(b)$. We want:

$$\int \frac{a}{b}$$

First Idea

Factor b over its splitting field K_b:

$$b = \prod_{i=1}^{m} (x - \beta_i)$$

Partial Fraction Expansion:

$$\frac{a}{b} = \sum_{i=1}^{m} \frac{\gamma_i}{x - \beta_i} \text{ where } \gamma_i, \beta_i \in K_b$$

Problem:

for $\deg(b) = m \rightarrow$ worst case degree of K_b over K is $m!$

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Get:

$$\int \frac{a}{b} = \sum_{i=1}^{m} \gamma_i \cdot \log(x - \beta_i)$$

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Rothstein/Trager Method (rational function case)

Theorem

For $a, b \in K[x]$ as before the minimal algebraic field extension necessary to express

$$\int \frac{a}{b}$$

is $K^* = K(c_1, c_2, \dots, c_n)$ where the c_i are the distinct roots of

$$R(z) = \mathrm{res}_x(a - zb', b) \in K[z].$$

Given K^*, c_i $(1 \le i \le n)$ as above

$$\int \frac{a}{b} = \sum_{i=1}^{n} c_i \cdot \log(v_i)$$

with

$$v_i = \gcd(a - c_i b', b) \in K^*[x].$$

Reminder: Resultant

Definition (Resultant)

For $a(x) = \sum_{i=0}^{m} a_i x^i, b(x) = \sum_{i=0}^{n} b_i x^i \in R[x]$

$$\operatorname{res}_x(a,b) := \left| \begin{pmatrix} a_m & a_{m-1} & \cdots & a_1 & a_0 & & \\ & a_m & a_{m-1} & \cdots & a_1 & a_0 & \\ & & \cdots & \cdots & \cdots & \cdots & \\ & & & a_m & \cdots & \cdots & a_0 \\ b_n & b_{n-1} & \cdots & b_1 & b_0 & & \\ & b_n & b_{n-1} & \cdots & b_1 & b_0 & \\ & & \cdots & \cdots & \cdots & \cdots & \\ & & & b_n & \cdots & \cdots & b_0 \end{pmatrix} \right|$$

$\operatorname{res}_x(0,b) := 0$ for $b \in R[x] \setminus \{0\}$, $\operatorname{res}_x(a,b) := 1$ for $a, b \in R \setminus \{0\}$.

Non-rational Functions

What about

$$\int \frac{1}{\exp(x) + 1}$$

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$$\int \frac{x}{\exp(x) + 1}?$$

Obviously these are not rational...

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What is an Elementary Function?

Definition

Let F be a differential field, G a differential extension field of F

1. $\theta \in G$ is called *logarithmic over* F, if $\exists\, u \in F$ such that
$$\theta' = \frac{u'}{u}.$$
Write $\theta = \log(u)$.
2. $\theta \in G$ is called *exponential over* F, if $\exists\, u \in F$ such that
$$\frac{\theta'}{\theta} = u'.$$
Write $\theta = \exp(u)$.
3. $\theta \in G$ is called *algebraic* over F, if $\exists\, p \in F[z] \setminus \{0\}$ such that
$$p(\theta) = 0.$$

Finding a Pattern

Examples of elementary functions and their integrals:

$$\int \cos(x) = \sin(x);$$
$$\int \frac{1}{\sqrt{1-x^2}} = \arcsin(x);$$
$$\int \operatorname{arccosh}(x) = x \operatorname{arccosh}(x) - \sqrt{x^2-1}.$$

Finding a Pattern

Examples of elementary functions and their integrals:

$$\int \left(\frac{1}{2}\exp(ix) + \frac{1}{2}\exp(-ix)\right) = -\frac{1}{2}i\exp(ix) + \frac{1}{2}i\exp(-ix);$$

$$\int \frac{1}{\sqrt{1-x^2}} = -i\log\left(\sqrt{1-x^2} + ix\right);$$

$$\int \log\left(x + \sqrt{x^2-1}\right) = x\log\left(x + \sqrt{x^2-1}\right) - \sqrt{x^2-1}.$$

Liouville's Principle

Theorem (Liouville)

Let F be a differential field, G an elementary extension field of F, K their common constant field.

$$g' = f$$

has a solution $g \in G$ if and only if there exist $v_0, v_1, \ldots, v_m \in F$, $c_1, \ldots, c_m \in K$ such that

$$f = v_0' + \sum_{i=1}^{m} c_i \frac{v_i'}{v_i}.$$

In other words, such that

$$\int f = v_0 + \sum_{i=1}^{m} c_i \log(v_i).$$

Proof - The rough Idea

- proof by induction on the number of new elementary extensions required to express the integral
- three cases: logarithmic, exponential or algebraic extensions
- basic arguments like polynomial arithmetic and differentiation
- for more details see: [Ros72] or [Ged92] pp. 523f

The Risch Algorithm - Logarithmic Case

Theorem (Rothstein/Trager Method - Logaritmic Case)

Let θ be transcendental and logarithmic over F (i.e. $\exists\, u \in F: \theta' = u'/u$); $a(\theta)/b(\theta) \in F(\theta)$ with $\gcd(a, b) = 1$, b monic and square-free.

$\int \frac{a(\theta)}{b(\theta)}$ is elementary if and only if all the roots of

$$R(z) = \text{res}_\theta(a(\theta) - z \cdot b(\theta)', b(\theta)) \in F[z]$$

are constants.

The Risch Algorithm - Logarithmic Case (cont'd)

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Let θ be transcendental and logarithmic over F (i.e. $\exists\, u \in F$: $\theta' = u'/u$); $a(\theta)/b(\theta) \in F(\theta)$ with $\gcd(a, b) = 1$, b monic and square-free.

If $\int \frac{a(\theta)}{b(\theta)}$ is elementary then

$$\frac{a(\theta)}{b(\theta)} = \sum_{i=1}^{m} c_i \frac{v_i(\theta)'}{v_i(\theta)}$$

where c_i are the distinct roots of $R(z)$ and

$$v_i(\theta) = \gcd(a(\theta) - c_i \cdot b(\theta)', b(\theta)) \in F(c_1, \ldots, c_m)[\theta].$$

The Risch Algorithm - Exponential Case

Theorem (Rothstein/Trager Method - Exponential Case)

Let θ *be transcendental and exponential over* F *(i.e.* $\exists\, u \in F$*:* $\theta'/\theta = u$*);* $a(\theta)/b(\theta) \in F(\theta)$ *with* $\gcd(a, b) = 1$*,* b *monic and square-free.*

$\int \frac{a(\theta)}{b(\theta)}$ *is elementary if and only if all the roots of*

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The Risch Algorithm - Exponential Case (cont'd)

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If $\int \frac{a(\theta)}{b(\theta)}$ *is elementary then*

$$\frac{a(\theta)}{b(\theta)} = g' + \sum_{i=1}^{m} c_i \frac{v_i(\theta)'}{v_i(\theta)}$$

where c_i *are the distinct roots of* $R(z)$*,*

$$v_i(\theta) = \gcd(a(\theta) - c_i \cdot b(\theta)', b(\theta)) \in F(c_1, \ldots, c_m)[\theta],$$

$$g' = -\left(\sum_{i=1}^{m} c_i \deg(v_i(\theta))\right) u'.$$

The Risch Algorithm - Algebraic Case

Surprise:

Algebraic case more complicated than transcendental cases!

- Liouville's Principle still holds
- algorithm for integral based on computational algebraic geometry
- for further details see:
  B. Trager "'Integration of Algebraic Functions"', Dept. of EECS, M.I.T. (1984)

Upper Triangular Systems

Definition (Upper Triangular System of ODEs)

Let K be a differential field and $p_{ij}(t) \in K$, $g_i(t) \in K$ $(1 \leq i \leq n)$.

$$\begin{cases} x_1'(t) = & p_{11}(t)x_1(t) \; + \; p_{12}(t)x_2(t) + \cdots + p_{1n}(t)x_n(t) + g_1(t), \\ x_2'(t) = & p_{22}(t)x_2(t) + \cdots + p_{2n}(t)x_n(t) + g_2(t), \\ \vdots & \vdots \\ x_n'(t) = & p_{nn}(t)x_n(t) + g_n(t) \end{cases}$$

is upper triangular system with initial conditions

$$x_1(0) = a_1, \quad x_2(0) = a_2, \ldots \quad , \; x_n(0) = a_n.$$

p_{ij} continuous for $t \in (a, b) \rightarrow$ unique solution for $t \in (a, b)$

Integrating Factor

Use back substitution to solve system!

$$x_n'(t) = p_{nn}(t)x_n(t) + g_n(t)$$

Integrating Factor

Multiply both sides by

$$\mu(t) := \exp\left(-\int p_{nn}(t)dt\right)$$

to get

$$x_n(t) = \frac{1}{\mu(t)}\left(\int \mu(t)g_n(t)dt + C_n\right)$$

C_n is chosen to satisfy the initial condition.

Solving the System by Recursion

- Substitute $x_n(t)$ into the equation for $x_{n-1}(t)$:

$$x'_{n-1}(t) = p_{n-1n-1}(t)x_{n-1}(t) + p_{n-1n}(t)x_n(t) + g_{n-1}(t)$$

- New integrating factor:

$$\exp\left(-\int p_{n-1n-1}(t)dt\right)$$

- Continue recursivly until all x_i are known

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Thank you for your attention!